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Structure and evolution of a spherical dust star. 1. The modified Oppenheimer-Snyder solution

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Abstract

In the Oppenheimer-Snyder solution (OS) for the parabolic trajectory particle's worldline r(t,R) in terms of world time t differs from its standard worldline in the Schwarzschild field outside and on the surface of the dust star. This is a consequence of the fact that the trajectory function r(t,R) were defined on the "homogeneity hypersurface", when r = R at the zero initial moment of proper time in all layers and since these events are not simultaneous, the initial moments of the world time t(R) are nonzero. In view of the fact that the structure of the star at any moment means the determination of the positions of all particles on the hypersurface t=const., and the solution of the OS is used for checking more realistic models of stars, this incompleteness of the procedure for the transition to hypersurface t=const. leads to distortions of physical consequences other models too. A more consistent application of the OS method is proposed, where this problem does not arise. The modification consists in fixing the initial positions r=R for t(R)=0 and determining the shift of the proper time moments in different layers on the hypersurfaces t=const. from the condition of obtaining the standard trajectory function r(t,R). The pictures of particle trajectories of the dust star are presented, which clearly show the internal structure of the star at t=const. At large t, not only the surface asymptotically approaches the gravitational radius, but the world lines of particles in the inner layers also approach their asymptotes, rapidly becoming practically parallel to the world lines of particles at the center and on the surface. This shows that the frozen star picture refers not only to the surface, but also to the inner layers freezing at certain distances from the center.

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Content

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Introduction

The internal structure of collapsing stars of mass $M > 3M_{\odot}$, when gravity dominates, is one of main problems of relativistic astrophysics [1-8]. In general relativity (GR), the picture of compression is complicated by the effects of relativistic kinematics and relativistic gravity. Nevertheless, for a spherical star with homogeneous dust matter and with a diagonal metric, this problem occurred to be exactly solvable [2-5], and thus the structure of the such star was considered as well-known [6-8].

However, in reality almost all the collapse models of such a star included some local coordinate transformations, which made it difficult or impossible to determine the structure of the star as a whole [4-8]. The point is that the definition of the structure of the star "at a given moment" means the assignment of the positions of all particles of the star as a collection of simultaneous events defined on a global hypersurface of simultaneity. For a diagonal metric such hypersurfaces can be chosen in many ways, but each of them must preserve the original symmetries and simplifying assumptions, which allow one to find exact solutions. In addition, on the surface of the star, the internal metric must be matched by the external Schwarzschild metric.

In practice, it turned out that in this case it is difficult to coincide the initial simplifying assumptions with the conditions for determination of the structure of the star as a whole. In particular, exact solutions can be found in local comoving coordinates $(\tau, R, \theta, \varphi)$ at describing the events on "hypersurfaces of homogeneity", where densities and proper times ("hypersurfaces of the same age") are the same in all layers. However, the velocities of these layers are different and therefore they do not have a single hypersurface of simultaneity. Thus, in the comoving coordinates, the structure of the star as a whole turned out to be undefined.

If, on the other hand, the metric in the comoving coordinates is transformed into a metric in the Schwarzschild coordinates (t, r, θ, φ) given on a global hypersurface of simultaneity $t = const.$, which simplifies the matching on the surface with the external static metric, then this metric is either given in different layers at different moments of world time [5-7] or nondiagonal, and a complicated diagonalization procedure was required [8]. In both cases there is again no single hypersurface of simultaneity - either because of local shifts in the initial world time, or local time transformations during diagonalization. In particular, in the Oppenheimer-Snyder solution (OS) for the parabolic velocity [5], the trajectory function $r(t, R)$ outside and on the surface of the star differs from the standard trajectory function in the Schwarzschild field, which is inadmissible and testifies incompleteness of the procedure for transforming of the exact solutions to the coordinates r,t.

Thus, the obvious task of determining the instantaneous structure of the dust star in the framework of GR appears as a nontrivial problem, requiring more careful analysis and taking into account the physical conditions of the collapse.

Since the OS model is used as a standard normalization model, on which both analytical and numerical models of relativistic stars are tested, the consequences of inaccuracies in this model go far beyond itself and distort also the physical consequences of more realistic models of relativistic astrophysics.

The aim of the present paper is to improve the OS model by a more consistent application of the OS method to achieve the consistency of a diagonality condition for the metric with the correct trajectory function of the particle in the Schwarzschild field on the hypersurface $t = const$.

In the original OS method [5], the solutions $r(\tau, R)$ are taken for events on "homogeneity hypersurfaces", where the initial moments of proper time τ in different layers are the same and $r(0,R) = R$ for $\tau_0(R) = 0$. But these events are not simultaneous and $t_0(R) \neq 0$, so the definition of the structure of the star as a whole is questionable. The proposed modification of the OS method consists in considering the such solutions $r(t,R)$ on the hypersurface of simultaneity, which follow from the choice $r(0,R) = R$ at $t_0(R) = 0$. Then the initial moments of proper time in different layers have shifts $\tau_0(R) \neq 0$ and they can be determined from the condition of obtaining the standard trajectory function in the Schwarzschild field. Thus, if the diagonality of the metric allows us to find the exact solution of the problem, then taking simultaneous events on the physical trajectories of the particles makes it possible to determine the structure of the dust star as a whole at any moment $t = const$.

In the Section 1 main relations for particle trajectories in the Schwarzschild field and the Tolman solution for the dust ball are presented. In Section 2 the OS method and solutions are described. In Section 3 the problem of time shift in the equations of trajectories is discusses and the modification of the OS method, in which this problem is not present, is describes. In Section 4 the structure and evolution of a dust star are studied, a picture of world lines is presented in the initial and modified OS models. In Appendix the derivation of the OS ansatz is presented.

1. Trajectories in the Schwarzschild field and Tolman's solution for the dust ball

1.1. Trajectories of particles in the Schwarzschild field

The space-time interval outside and on the surface of the spherical star $r \geq r_b$ (where r_b is the circumferential radius of the surface) in static coordinates is given by the Schwarzschild solution [1]:

$$ds^2 = \alpha dt^2 - \alpha^{-1} dr^2 - r^2 d^2\Omega, \tag{1}$$

where $\alpha \equiv 1 - r_g / r$, $r_g = 2GM$, and $d^2\Omega = d\theta^2 + \sin^2\theta d\varphi^2$.

Let us consider the radial falling of test particles in this field with initial coordinates $r(t_0) = R > R_b$ at time $t_0 = 0$. Their local velocities have the form:

$$v^2 = \frac{1}{(1 - r_g / r)^2} \frac{dr^2}{dt^2}. \tag{2}$$

In the static field, the energy of particle is conserved and at a parabolic velocity, when the velocity of free fall is such that it would be zero at spatial infinity, the energy conservation condition gives:

$$\frac{1-r_g/r}{1-v^2}=1, \quad v=-r_g^{1/2}/r^{1/2}. \tag{3}$$

From (1)-(3) we find for the proper time interval of the particles:

$$d\tau^2=\frac{dr^2}{1-r_g/r}\left(\frac{1}{v^2}-1\right)=\frac{r}{r_g}dr^2, \tag{4}$$

which gives:

$$\tau-\tau_0(R)=\frac{2}{3r_g^{1/2}}(R^{3/2}-r^{3/2}). \tag{5}$$

The trajectory function $r(\tau,R)$ thus has the form:

$$r(\tau,R)=\left(R^{3/2}-\frac{3}{2}r_g^{1/2}[\tau-\tau_0(R)]\right)^{2/3}. \tag{6}$$

For the world time interval t from (2) and (3) follows

$$t=-\frac{1}{r_g^{1/2}}\int_R^{r(t)}\frac{dr\,r^{3/2}}{r-r_g}=\frac{1}{r_g^{1/2}}\int_{r(t)}^{R}\frac{dr\,r^{3/2}}{r-r_g} \tag{7}$$

and the integration gives:

$$t=\frac{2}{3r_g^{1/2}}(R^{3/2}-r^{3/2})+2r_g^{1/2}(R^{1/2}-r^{1/2})+r_g\ln\left(\frac{r^{1/2}+r_g^{1/2}}{r^{1/2}-r_g^{1/2}}\cdot\frac{R^{1/2}-r_g^{1/2}}{R^{1/2}+r_g^{1/2}}\right). \tag{8}$$

The expression (8) gives the trajectory function $r(t,R)$ in the implicit form.

The times τ and t are two parametrizations of the same events along the world line of the particle $r(\tau,R)=r(t,R)$, thus these two times are mutually related and irreversible dilation of τ w.r.t. t is the experimental fact (see [9]). This connection $\tau(t,R)$ in the implicit form follows from (6) and (8):

$$t=\tau-\tau_0(R)+2r_g^{1/2}[R^{1/2}-r^{1/2}(\tau,R)]+r_g\ln\left(\frac{r^{1/2}(\tau,R)+r_g^{1/2}}{r^{1/2}(\tau,R)-r_g^{1/2}}\cdot\frac{R^{1/2}-r_g^{1/2}}{R^{1/2}+r_g^{1/2}}\right). \tag{9}$$

The plot of the relation of two times $\tau(t,R)$ is shown in Fig. 1. The points on them with $t=const.$ correspond to simultaneous events, and with $\tau=const.$ - “the same age” ones. In the plots of the trajectories $r(t,R)$ and $r(\tau,R)$ (Fig. 2), it is clear that there is a one-to-one correspondence between the events on both curves $r(t,R)=r(\tau,R)$, and therefore both curves are asymptotic, which means that at $t\to\infty$ both asymptotically approach the gravitational radius $r(t,R)\to r_g$ and $r[\tau(t,R)]\to r_g$. At $t\to\infty$ and $r\to r_g$ the freezing moments of proper times $\tau\to\tau_g$ we find from (5):

$$\tau_g(R)=\tau_0(R)+\frac{2}{3r_g^{1/2}}(R^{3/2}-r_g^{3/2}). \tag{10}$$

Density of test particles, if they were initially distributed uniformly, in the future will also remain homogeneous at “the same age” hypersurface, which is why it is called as the “hypersurface of homogeneity”. Points of intersection of lines of “one-age” with world lines are given by the expression (9).

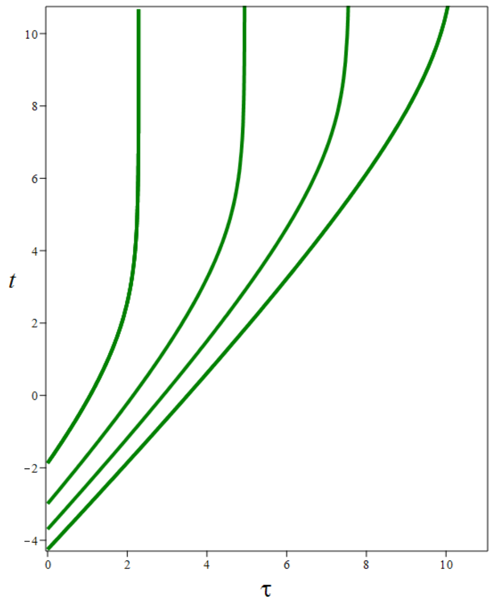

Fig.1. The proper times of test particles $\tau = \tau[r(t)] = \tau(t)$ which were at $R = 2,3,4,5$ (in units r_g) at $t = 0$, asymptotically freeze at $t \gg r_g$ (see [9]).

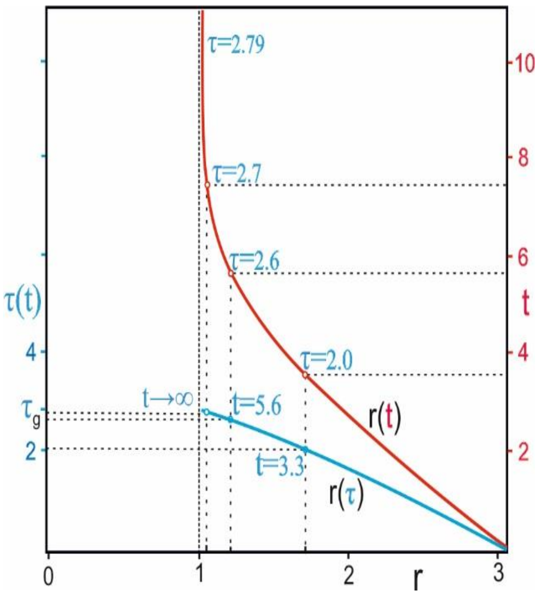

Fig. 2. The world line of a falling test particle in terms of world time *t* and proper time τ .

1.2. The general solution of Tolman for the dust ball in the comoving coordinates

Inside the dust star ($r < r_b$) the linear element in the comoving coordinates can be written as:

$$ds^2 = c^2 d\tau^2 - e^{\lambda(\tau,R)} dR^2 - r^2(\tau,R) d\Omega^2. \qquad (11)$$

Here $2\pi r(\tau,R)$ is the length of the circle around the star.

Exact solutions of Einstein's equations with such a metric were found in general form by Tolman [4]. They contain two unknown functions of the initial coordinates of the layers $f(R)$ and $F(R)$ (in the notations of [7]) and, therefore, to find exact solutions it was necessary to specify the physical conditions for the contracting dust. For our purposes, these are homogeneity and parabolic velocities.

When the partial derivative under τ is denoted by a point, and under R by the prime, Tolman's solution has the form:

$$e^{\lambda} = r'^2/(1+f), \quad \dot{r}^2 = f + F/r, \quad 8\pi G\rho = \frac{F'}{r'r^2}. \qquad (12)$$

Here $1+f>0$, G is the gravitational constant, ρ is the energy density of the dust matter. Integration in (12) gives:

$$\tau = \pm \int \frac{dr}{(f + F/r)^{1/2}}. \qquad (13)$$

For parabolic motion $f = 0$ and we obtain an analogue of the expression for the proper time (5):

$$\tau - \tau_0(R) = \frac{2}{3F^{1/2}}(R^{3/2} - r^{3/2}). \tag{14}$$

Thus, the linear element in Tolman's solution takes the form:

$$ds^2 = c^2 d\tau^2 - r'^2 dR^2 - r^2(\tau, R) d\Omega^2. \tag{15}$$

2. The OS solution: a homogeneous spherical ball with the parabolic velocity

2.1. Metric and trajectories in the external region

Oppenheimer and Snyder [5] have studied the Tolman solution for the special case when: a) the dust star is homogeneous in the comoving coordinates, and b) the dust particles fall with the parabolic velocity and $f(R) = 0$. By transforming Tolman solutions to the coordinates r, t, they were able to find exact solutions in these coordinates to internal metrics and trajectories and sew them on the surface with the external Schwarzschild metric.

Tolman's solutions inside the homogeneous star in the comoving coordinates naturally coincide with Friedmann's solutions for the flat case previously known from cosmology. Later, other authors used Friedmann's solutions for positive and negative curvature also, corresponding to elliptic ($f < 0$) and hyperbolic ($f > 0$) velocities. In the literature, both they are also referred to as OS solutions [6-8] in view of the homogeneity condition used by the OS in obtaining the solution for the parabolic case.

Let us consider the test particles outside the star $r > r_b$ falling with parabolic velocities. Since the metric in the external region is known and given by the Schwarzschild solution (1), we proceed from it and transform it from the coordinates r,t into the R, τ coordinates.

We express the intervals r,t in terms of intervals of R, τ in the form:

$$dt = \dot{t}\, d\tau + t' dR, \quad dr = \dot{r}\, d\tau + r' dR. \tag{16}$$

Thus the linear element (1) passes into:

$$\begin{aligned} ds^2 &= \alpha dt^2 - \alpha^{-1} dr^2 - r^2 d^2\Omega = \\ &= (\alpha \dot{t}^2 - \alpha^{-1}\dot{r}^2) d\tau^2 - (\alpha^{-1} r'^2 - \alpha t'^2) dR^2 + \\ &+ 2(\alpha \dot{t}\, t' - \alpha^{-1} \dot{r} r') d\tau dR - r^2(\tau, R) d^2\Omega. \end{aligned} \tag{17}$$

Expressions for the components of the metric in the coordinates R, τ are also known from (15) and together with (17) they give two equations for determining $\dot{t}$, t', as well as the diagonality condition for the metric:

$$g_{00}(\tau, R) = \alpha \dot{t}^2 - \alpha^{-1} \dot{r}^2 = 1, \tag{18}$$

$$g_{11}(\tau, R) = -(\alpha^{-1} r'^2 - \alpha t'^2) = -r'^2, \tag{19}$$

$$g_{01}(\tau, R) = \alpha \dot{t}\, t' - \alpha^{-1} \dot{r} r' = 0. \tag{20}$$

Taking into account $F = r_g$, from (3), (12), (18) and (19), we find $\dot{r}$, $\dot{t}$ and t':

$$\dot{r} = -r_g^{1/2} / r^{1/2}, \quad \dot{t} = \alpha^{-1}, \quad t'^2 = r'^2 \alpha^{-2} r_g / r. \tag{21}$$

In the paper [5], in the expression (14) the special case with $\tau_0(R) = 0$ (the hypersurface of one-age) is chosen and the equation of the trajectory is taken in the form:

$$r(\tau, R) = (R^{3/2} - 3 r_g^{1/2} \tau / 2)^{2/3}. \tag{22}$$

This is followed by the expressions for r' and t':

$$r' = \frac{R^{1/2}}{r^{1/2}}, \quad t' = -\frac{(r_g R)^{1/2}}{r - r_g}. \tag{23}$$

The condition of diagonality of the metric (20) as a result takes the form:

$$\frac{t'}{\dot{t}} = r'\dot{r} = -\frac{(r_g R)^{1/2}}{r}. \tag{24}$$

This condition, however, turned out to be incompatible with the equation of the trajectory in the Schwarzschild field (8), and in the paper [5] another equation of the trajectory was used which agrees with (24):

$$t = \frac{2}{3r_g^{1/2}}(R^{3/2} - r^{3/2}) - 2r_g^{1/2} r^{1/2} + r_g \ln\frac{r^{1/2} + r_g^{1/2}}{r^{1/2} - r_g^{1/2}}. \tag{25}$$

It follows from (22) that $r = R$ at $\tau = 0$ and then (25) gives:

$$t_0(R) = -2r_g^{1/2} R^{1/2} - r_g \ln\frac{R^{1/2} - r_g^{1/2}}{R^{1/2} + r_g^{1/2}}, \tag{26}$$

i.e. when $r = R$ in different layers the world time has a different values $t_0(R) \neq 0$. Thus, in the original OS method [5] as initial ones have been taken the events with $\tau_0(R) = 0$ and as a result the equation of trajectory (25), in contrast to (9), contains shifts $t_0(R) \neq 0$.

2.2. Metrics and trajectories in the internal region

In the previous section, the known external solutions in terms of r,t have been transformed to the solutions in terms of R,τ. Here, following [5], we will perform the inverse transformation - from Tolman's internal solutions in terms of R,τ we will obtain a solution in terms r,t. This is necessary both for determining the instantaneous structure of the star as a whole, and for matching with the external metric (1).

The linear element inside the star with the desired metric has the form:

$$ds^2 = e^{\nu(r,t)}dt^2 - e^{\lambda(r,t)}dr^2 - r^2 d^2\Omega, \quad r \le r_b \tag{27}$$

and matching on the surface should give $e^{\nu(r_b,t)} = e^{-\lambda(r_b,t)} = 1 - r_g / r_b$. It follows from (16) that:

$$\begin{aligned} ds^2 &= e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2 d^2\Omega = \\ &= (e^{\nu}\dot{t}^2 - e^{\lambda}\dot{r}^2)d\tau^2 - (e^{\lambda}r'^2 - e^{\nu}t'^2)dR^2 + 2(e^{\nu}\dot{t}t' - e^{\lambda}\dot{r}r')d\tau dR - r^2 d^2\Omega. \end{aligned} \tag{28}$$

Assuming $f = 0$, as in the external region, from the correspondence with (15) we obtain the conditions:

$$g_{00}(\tau, R) = e^{\nu}\dot{t}^2 - e^{\lambda}\dot{r}^2 = 1, \tag{29}$$

$$g_{11}(\tau, R) = -(e^{\lambda}r'^2 - e^{\nu}t'^2) = -r'^2, \tag{30}$$

$$g_{01}(\tau, R) = e^{\nu}\dot{t}t' - e^{\lambda}\dot{r}r' = 0. \tag{31}$$

The first two of them give:

$$e^{\lambda} = \frac{\dot{t}^2 r'^2 + t'^2}{\dot{t}^2 r'^2 - t'^2 \dot{r}^2}, \qquad e^{\nu} = \frac{r'^2(1 + \dot{r}^2)}{\dot{t}^2 r'^2 - t'^2 \dot{r}^2}, \tag{32}$$

and the third one (31) is the condition of diagonality, which, taking into account (31), takes the form:

$$g_{01}=\frac{\dot{t}t'r'^2(1+\dot{r}^2)-(\dot{t}^2r'^2+t'^2)\dot{r}r'}{\dot{t}^2r'^2-t'^2\dot{r}^2}=0. \tag{33}$$

Writing the numerator (33) as an equation for t':

$$\dot{r}t'^2-[\dot{t}r'(1+\dot{r}^2)]t'+\dot{t}^2r'^2=0, \tag{34}$$

we obtain two solutions

$$t'=\frac{\dot{t}r'}{2\dot{r}}(1+\dot{r}^2)\pm\frac{\dot{t}r'}{2\dot{r}}(1-\dot{r}^2). \tag{35}$$

The first of them $t'=\dot{t}r'/\dot{r}$ is unphysical, since it diverges at $\dot{r}\to 0$, and the second gives the diagonality condition the same as in the external region:

$$t'/\dot{t}=r'\dot{r}. \tag{36}$$

Excluding t' by it from (32), we get a solution for the metric:

$$e^{\lambda}=\frac{1}{1-\dot{r}^2},\quad e^{\nu}=\frac{1}{\dot{t}^2(1-\dot{r}^2)}. \tag{37}$$

The homogeneity condition gives for the function $F(R)$:

$$F(R)=r_g\frac{R^3}{R_b^3}\equiv r_{g,R}. \tag{38}$$

As in the case of outer region, in [5] the expression for τ has been chosen at the condition $\tau_0(R)=0$:

$$\tau=\frac{2}{3r_{g,R}^{1/2}}(R^{3/2}-r^{3/2}). \tag{39}$$

The trajectory function $r(\tau,R)$ thus takes the form:

$$r(\tau,R)=R\left(1-\frac{3r_g^{1/2}}{2R_b^{3/2}}\tau\right)^{2/3}. \tag{40}$$

from which we find the expressions for $\dot{r}$ and r':

$$\dot{r}=-\frac{r_{g,R}^{1/2}}{r^{1/2}},\quad r'=\frac{r}{R}. \tag{41}$$

To determine the trajectory function $r(t,R)$ in the OS method [5], it has been introduced a new variable y:

$$t=M(y), \tag{42}$$

where the form of the function M is determined from the condition of matching with the external metric. The diagonality condition of the metric (36) then takes the form:

$$\frac{t'}{\dot{t}}=\frac{y'}{\dot{y}}=\dot{r}r'=-\frac{(r_{g,R}r)^{1/2}}{R}. \tag{43}$$

The solution of this equation with respect to y, proposed by OS [5], is reproduced in the Appendix and the result is:

$$y(R,\tau)=\frac{1}{2}\left(\frac{R^2}{R_b^2}-1\right)+\frac{R_b}{Rr_g}r. \tag{44}$$

Here $y_b=y(R_b)=R_b/r_g$ and on the surface the function $M(y_b)$ must coincide with the right-hand side of (25).

Making in Eq.(25) the substitutions $R \to R_b$ and $r = r_g y_b$, it has been obtained in [5] the trajectory function $r(t,R)$ for the particles of the dust star in the implicit form:

$$t = \frac{2}{3r_g^{1/2}}(R_b^{3/2} - r_g^{3/2} y^{3/2}) - 2r_g y^{1/2} + r_g \ln \frac{y^{1/2}+1}{y^{1/2}-1}. \qquad (45)$$

For $\dot{t}$ and t' this relation, taking into account $y' = R/R_b^2$ and $\dot{y} = -(R/rr_g R_b)^{1/2}$, gives:

$$\dot{t} = -\frac{r_g y^{3/2} \dot{y}}{y-1} = \left(\frac{r_g R}{R_b r}\right)^{1/2} \frac{y^{3/2}}{y-1}, \qquad (46)$$

$$t' = -\frac{r_g y^{3/2} y'}{y-1} = -\frac{r_g R}{R_b^2} \frac{y^{3/2}}{y-1} \qquad (47)$$

Thus, the OS method consists in transforming Tolman's solution from the comoving coordinates R,τ to the Schwarzschild coordinates r,t and then matching on the surface of the internal metric with the external. The metric (37), together with the equations of the trajectory (40)-(47), is the internal OS solution.

3. The modified OS method

3.1. The time shift problem in the trajectory function in the OS method

The OS solution, given above in detail, is based on the choice for $r = R$ in (39) zero initial moments of proper times of all layers $\tau_0(R) = 0$. But then the initial values of the positions of the particles R in these layers are not given simultaneously, since $t_0(R) \neq 0$ in (25). The physical meaning of this fact is that in the OS paper [5] Tolman's solution, given in the comoving coordinates on the hypersurface of "one age" $\tau = const.$, is transformed into the Schwarzschild coordinates r,t on the same hypersurface.

As a result, the solution of the OS in terms of r,t at different layers turned out to be not specified on the hypersurface of simultaneity $t = const.$ Instead of simultaneous events with the same t, there are taken events containing a shift $t_0(R)$ from (26) in the external region and a shift $t_0(R_b)$ inside, where:

$$t_0(R_b) = -2r_g^{1/2} R_b^{1/2} - r_g \ln \frac{R_b^{1/2} - r_g^{1/2}}{R_b^{1/2} + r_g^{1/2}}. \qquad (48)$$

At the same time, the external Schwarzschild metric is defined at $t = const.$, where the initial positions $r = R$ are given for $t_0(R) = 0$. In addition, the very definition of the instantaneous structure of the star as a whole presupposes the fixing of the solutions for different layers on this hypersurface only. But in the OS method this leads to a nondiagonal metric with which Einstein's equations become practically unsolvable.

In practice, this reveals in the form of a difference in the trajectories obtained from the OS analytical solution from the trajectories of real particles obtained by the numerical solutions with the correct physical conditions. This is the time shift problem for the trajectory function in the OS method.

Thus, although the OS method was a big step in the right direction, however, in this issue it is not brought to its logical conclusion. The procedure for obtaining the exact solution of the problem was interrupted and the OS method lacks the final steps.

Below the required modification of the OS method with such final steps will be presented, when starting from the standard trajectory function (8) and obeying the condition of diagonality of the metric, it will be obtained a solution of the problem by correctly describing the star's structure on the hypersurface of simultaneity $t = const$.

3.2. Modification of the OS method for the external region

A consecutive transition to the coordinates r,t assumes the description of events along the particle worldlines in all layers $r(t,R)$ on the hypersurface $t = const$. Therefore, the first step in the modification of the OS method is the transition to this hypersurface, which means $r = R$ at $t_0(R) = 0$ with the choice of the trajectory equation in the standard form (8). However, in the trajectory equation (5)-(6) there appear the local shifts $\tau_0(R) \neq 0$, which changes the value r' in comparison with (23):

$$r' = \frac{1}{r^{1/2}}(R^{1/2} + r_g^{1/2}\tau_0'). \tag{49}$$

The derivatives $\dot{r}$ and $\dot{t}$ are still given by (21), but t', following from the the trajectory function (8), now has another form:

$$t' = \frac{R^{3/2}}{r_g^{1/2}(R - r_g)} - \frac{r^{3/2} r'}{r_g^{1/2}(r - r_g)}. \tag{50}$$

The condition of diagonality, however, taking into account (21), gives:

$$t' = \dot{t}\dot{r}r' = -\frac{(r_g r)^{1/2}}{r - r_g} r'. \tag{51}$$

The first expression for t' in (50) follows from the trajectory function (8), and the second expression (51) - from the diagonality of the metric and for coinciding of these two physical conditions, both expressions must give the same value for t'.

At first look this seems problematic, since these two expressions differ sufficiently. However, now we remember that until we have not determined the value of r' in (49), since we have not yet clear up to what is equal $\tau_0(R)$. The second step in the modification of the OS method therefore is that we find r' and $\tau_0(R)$ from the consistency of the trajectory function and the diagonality condition.

For this, it is enough to equate (50) and (51), that gives an equation for the definition of r':

$$t' = \frac{R^{3/2}}{r_g^{1/2}(R - r_g)} - \frac{r^{3/2} r'}{r_g^{1/2}(r - r_g)} = -\frac{(r_g r)^{1/2}}{r - r_g} r'. \tag{52}$$

Solving this equation w.r.t. r' we obtain for it the required expression:

$$r' = \frac{R^{1/2}}{r^{1/2}} \frac{R}{R - r_g}, \tag{53}$$

which differs from (23). From (49) and (53) then follows the equation for $\tau_0(R)$ too:

$$\tau_0' = \frac{(r_g R)^{1/2}}{R - r_g} \tag{54}$$

and integrating of this expression yields:

$$\tau_0(R) = 2 r_g^{1/2} R^{1/2} + r_g \ln \frac{R^{1/2} - r_g^{1/2}}{R^{1/2} + r_g^{1/2}}. \tag{55}$$

Thus, outside the star the initial shifts $\tau_0(R)$ in the modified OS method turned out to be equal, but inverse to the shifts $t_0(R)$ of the OS method (26). From (55) and (10) then we obtain for the moments of freezing of proper times $\tau_g(R)$:

$$\tau_g(R) = \frac{2}{3r_g^{1/2}}(R^{3/2} - r_g^{3/2}) + 2r_g^{1/2}R^{1/2} + r_g \ln\frac{R^{1/2} - r_g^{1/2}}{R^{1/2} + r_g^{1/2}}. \qquad (56)$$

The equation of the trajectory (8), taking into account (5) and (55), then takes a more compact form:

$$t = \tau - 2r_g^{1/2}r^{1/2} + r_g \ln\frac{r^{1/2} + r_g^{1/2}}{r^{1/2} - r_g^{1/2}}. \qquad (57)$$

The derivative t' calculated from (57) is different from (23) and is now equal to:

$$t' = -\frac{(r_g R)^{1/2}}{r - r_g}\frac{R}{R - r_g}. \qquad (58)$$

Thus, the modification of the OS method is reduced to:

A) the choice $r = R$ at the initial time $t_0(R) = 0$ and $\tau_0(R) \neq 0$ as in (55), the inverse of the former method of OS, where it was $t_0(R) \neq 0$ and $\tau_0(R) = 0$,

B) the choice of the standard particle trajectory function in the Schwarzschild field (8) or (57), and

C) finding r' and $\tau_0(R)$ from the consistency of the trajectory function and the diagonality condition.

These final steps, added to the OS method, provide both the correct trajectory functions and the required changes in r' and t', necessary to satisfy the diagonality condition on the hypersurface of simultaneity $t = const$.

3.3. Modification of the OS method for the internal region

Inside the homogeneous dust star, the solution (14) in the original OS method it was find at the condition $\tau_0(R) = 0$ and it was based on the trajectory function (39)-(40).

In the modified OS method, the internal solution (14) should be taken with a constant shift $\tau_0(R_b) \neq 0$ on the star's surface, which allows one to sew it with the external solution (5):

$$\tau - \tau_0(R_b) = \frac{2}{3r_{g,R}^{1/2}}(R^{3/2} - r^{3/2}), \qquad (59)$$

where, according to (55), $\tau_0(R_b)$ is equal to:

$$\tau_0(R_b) = 2r_g^{1/2}R_b^{1/2} + r_g \ln\frac{R_b^{1/2} - r_g^{1/2}}{R_b^{1/2} + r_g^{1/2}}. \qquad (60)$$

Instead of the trajectory function of the OS (40), respectively, we have the trajectory function:

$$r(\tau, R) = R\left(1 - \frac{3r_g^{1/2}}{2R_b^{3/2}}[\tau - \tau_0(R_b)]\right)^{2/3}, \qquad (61)$$

so the derivatives $\dot{r}$ and r' will be the same as in (41), but now $\tau_0' = 0$.

In the trajectory function $r(t,R)$, according to the OS method, R should be replaced by R_b, and instead of r is substituted $r_g y$, which gives:

$$t=\frac{2}{3r_g^{1/2}}(R_b^{3/2}-r_g^{3/2}y^{3/2})+2r_g^{1/2}(R_b^{1/2}-r_g^{1/2}y^{1/2})+r_g\ln\left(\frac{y^{1/2}+1}{y^{1/2}-1}\cdot\frac{R_b^{1/2}-r_g^{1/2}}{R_b^{1/2}+r_g^{1/2}}\right). \quad (62)$$

In contrast to the Eq. (45) used in [5], this function on the surface is matched with the standard trajectory function in the Schwarzschild field (8).

Derivatives $\dot{t}$ and t' will be as before (46)-(47), so that the diagonality condition of the metric (43) will be satisfied in the modified OS method. The form of the expression for the internal metric in the OS solution (37) does not change, but these equations enter $r(t,R)$, obeying the equation (62), not (45), and, therefore, the values of the metric will already be different.

4. The internal structure and evolution of a dust star

4.1. Trajectories of particles in layers and their asymptotes

At $t\to\infty$ the surface asymptotically approaches the gravitational radius $r_b\to r_g$ and in (8) the growing logarithmic term dominates, which gives in the external region and on the surface:

$$t\simeq -r_g\ln\left(\frac{r_b^{1/2}-r_g^{1/2}}{r_b^{1/2}+r_g^{1/2}}\right). \quad (63)$$

$$e^{-t/r_g}\simeq\frac{r_b-r_g}{(r_b^{1/2}+r_g^{1/2})^2}\simeq\frac{1}{4}\left(\frac{r_b}{r_g}-1\right), \quad (64)$$

$$r_b(t)\simeq r_g(1+4e^{-t/r_g})>r_g. \quad (65)$$

In the trajectory functions (45) and (62) the asymptotic behavior on the surface is the same, since $y_b\to 1$ and it also gives (65).

In the inner layers, the growth of the logarithmic contribution also occurs at $y\to 1$ and this gives for the asymptotic behavior of the layers $r(t,R)\to r(\infty,R)$ the equation:

$$y(\infty,R)=\frac{1}{2}\left(\frac{R^2}{R_b^2}-1\right)+\frac{R_b}{r_g R}r(\infty,R)=1, \quad (66)$$

from which it follows:

$$r(\infty,R)=R\frac{3r_g}{2R_b}\left(1-\frac{R^2}{3R_b^2}\right)\sim R. \quad (67)$$

As we can see, the asymptotic lines of the layers at $t\to\infty$ are almost equidistant (Fig. 3).

The proper time on each layer freezes at a value $\tau_\infty(R)$ which we find by inserting $r(\infty, R)$ into (59) :

$$\tau_\infty(R) = \tau_0(R_b) + \\ + \frac{2R_b^{3/2}}{3r_g^{1/2}} - \frac{3^{1/2} r_g}{2^{1/2}}\left(1 - \frac{R^2}{3R_b^2}\right)^{3/2} \quad (68)$$

First of all freezes the center at the moment $\tau_\infty(0)$:

$$\tau_\infty(0) = \tau_0(R_b) + \\ + \frac{2}{3}\left(\frac{R_b^{3/2}}{r_g^{3/2}} - \frac{3^{3/2}}{2^{3/2}}\right) r_g \quad (69)$$

The surface freezes later than all layers at $\tau_\infty(R_b)$:

$$\tau_\infty(R_b) = \tau_0(R_b) + \\ + \frac{2}{3}\left(\frac{R_b^{3/2}}{r_g^{3/2}} - 1\right) r_g \quad (70)$$

The interval of proper time, during which all layers of the dust star are frozen, is equal to the difference between the moments of freezing of the surface and the center:

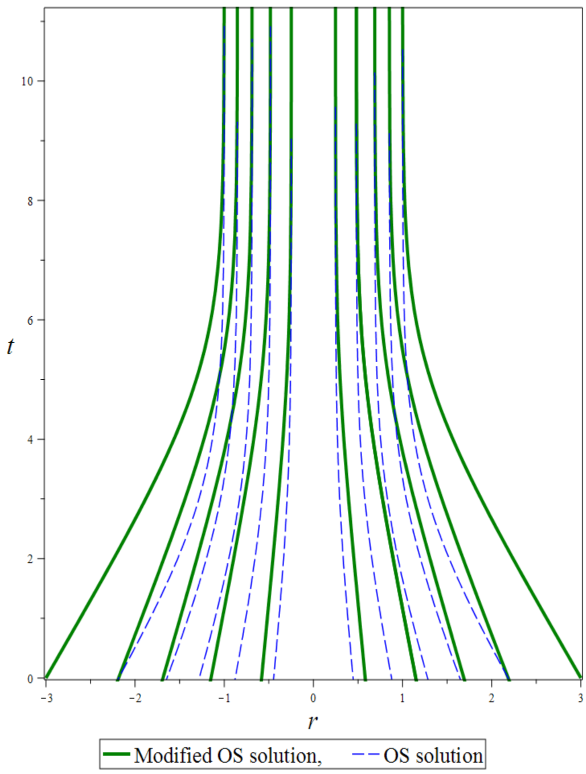

Fig. 3. The world lines of particles in the layers of the homogeneous dust ball (in units of r_g). The surface falls from R_b=3 at t = 0 and asymptotically approaches r_g at $t \to \infty$. Other layers freeze at different distances from the center according to (67).

$$\tau_\infty(R_b) - \tau_\infty(0) = \frac{2}{3}\left(\frac{3^{3/2}}{2^{3/2}} - 1\right) r_g \approx 0.558 r_g \quad (71)$$

Let us now consider the dependence on t of the basic variables at large t. From the asymptotes of t :

$$t \simeq -r_g \ln\left(\frac{y-1}{(y^{1/2}+1)^2}\right) \simeq -r_g \ln\left(\frac{y-1}{4}\right), \quad (72)$$

we obtain:

$$y = \frac{1}{2}\left(\frac{R^2}{R_b^2} - 1\right) + \frac{R_b}{R r_g} r \simeq 1 + 4e^{-t/r_g}, \quad (73)$$

$$r \simeq \frac{3 r_g R}{2R_b}\left(1 - \frac{R^2}{3R_b^2} + \frac{8}{3} e^{-t/r_g}\right). \quad (74)$$

By substituting this into (61), we then find the relationship between the moments of proper time inside the star $\tau(t, R)$ and t :

$$\tau = \tau_0(R_b) + \frac{2R_b^{3/2}}{3r_g^{1/2}} - \frac{3^{1/2} r_g}{2^{1/2}}\left(1 - \frac{R^2}{3R_b^2} + \frac{8}{3} e^{-t/r_g}\right)^{3/2}. \quad (75)$$

At $t \to \infty$ this expression goes into (68).

4.2. The intrinsic metric and its asymptotes

In the exact solution of the OS, the components of the metric (37), taking into account (41), (46) and (44), take the form:

$$e^{\lambda(t,R)} = (1 - r_{g,R} / r)^{-1}, \tag{76}$$

$$e^{\nu(t,R)} = \frac{[1 + (r_{g,R} / 2r)(1 - 3R_b^2 / R^2)]^2}{[1 + (r_{g,R} / 2r)(1 - R_b^2 / R^2)]^3} \frac{1}{1 - r_{g,R} / r}. \tag{77}$$

The line element, respectively, has the form:

$$ds^2 = \frac{[1 + (r_{g,R} / 2r)(1 - 3R_b^2 / R^2)]^2}{[1 + (r_{g,R} / 2r)(1 - R_b^2 / R^2)]^3} \frac{dt^2}{1 - r_{g,R} / r} - \frac{dr^2}{1 - r_{g,R} / r} - r^2 d^2\Omega. \tag{78}$$

On the surface $R = R_b$, this line element turns to the line Schwarzschild element (1). At the center of the star $R = r = 0$ the spatial metric is trivial $e^{\lambda(0,t)} = 1$.

At $t \gg r_g$ the metric (76), with the account (74), has the form:

$$e^{\lambda(t,R)} \simeq \left[1 - \frac{R^2}{R_b^2}\left(4e^{-t/r_g} + \frac{1}{2}\left(3 - \frac{R^2}{R_b^2}\right)\right)^{-1}\right]^{-1} \tag{79}$$

or in a more compact form:

$$e^{\lambda(t,R)} \simeq \frac{8e^{-t/r_g} + 3 - R^2 / R_b^2}{8e^{-t/r_g} + 3 - 3R^2 / R_b^2}. \tag{80}$$

At the center and on the surface its values are:

$$e^{\lambda(t,0)} = 1, \quad e^{\lambda(t,R_b)} \simeq e^{t/r_g} / 4. \tag{81}$$

At $t \to \infty$ the metric from (80) tends to:

$$e^{\lambda(\infty,R)} = \frac{1 - R^2 / 3R_b^2}{1 - R^2 / R_b^2}. \tag{82}$$

At $t \gg r_g$ the time component of the metric from (37), with taking into account (46) and (74), takes the form:

$$e^{\nu(t,R)} \simeq 16 e^{\lambda} e^{-2t/r_g}\left(4e^{-t/r_g} + \frac{1}{2}\left(3 - \frac{R^2}{R_b^2}\right)\right) \tag{83}$$

or in the expanded form:

$$e^{\nu(t,R)} \simeq e^{-2t/r_g} \frac{[8e^{-t/r_g} + (3 - R^2 / R_b^2)]^2}{e^{-t/r_g} + 3(1 - R^2 / R_b^2) / 8}. \tag{84}$$

At the center and on the surface its values are:

$$e^{\nu(t,0)} \simeq 24 e^{-2t/r_g}, \quad e^{\nu(t,R_b)} \simeq 4e^{-t/r_g}. \tag{85}$$

At $t \to \infty$ the metric from (84) tends to:

$$e^{\nu(\infty,R)} = 24 e^{-2t/r_g} \frac{(1 - R^2 / 3R_b^2)^2}{1 - R^2 / R_b^2}. \tag{86}$$

Conclusion

In the paper the collapse of the homogeneous dust star is studied and its instantaneous structure in terms of world time is described in the framework of the OS method [5] and its modification proposed in the paper.

The plots of the trajectories of dust star's particles (Fig. 3) clearly show the internal structure of the star on hypersurfaces of simultaneity. For large *t*, the worldlines of particles on the surface asymptotically approach the gravitational radius, and the worldlines in the inner layers, approaching own asymptotes, become practically parallel to the worldlines of the center and the surface both in the former and in the modified OS solutions.

This shows that the frozen star picture refers not only to the surface, freezing near r_g , but to the inner layers too, freezing near their asymptotes.

The remaining two classes of solutions for elliptic and hyperbolic velocities will be considered in forthcoming papers.

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Appendix

The solution of the equation (43):

$$\frac{y'}{\dot{y}} = -\frac{(r_{g,R}r)^{1/2}}{R} \tag{87}$$

we will look for among the functions of the form

$$y(R,r) = A(R) + B(R,r), \tag{88}$$

where A does not depend on r, and the partial derivative of B with respect to R is equal to zero:

$$y' = A' + B' + \frac{\partial B}{\partial r} r' = A'. \tag{89}$$

This gives

$$B' = -\frac{\partial B}{\partial r} r', \quad \dot{y} = \frac{\partial B}{\partial r}\dot{r}. \tag{90}$$

The diagonality condition (43) then takes the form:

$$\frac{y'}{\dot{y}} = A'\left(\frac{\partial B}{\partial r}\dot{r}\right)^{-1} = \dot{r}r' \tag{91}$$

from which we obtain:

$$A' = \dot{r}^2 r' \frac{\partial B}{\partial r} = \frac{r_{g,R}}{R}\frac{\partial B}{\partial r}. \tag{92}$$

Since the left-hand side does not depend on r, then the right-hand side should not contain r, which implies that B has the form $B = q(R)r$ and from (90) we obtain for q:

$$B' + \frac{\partial B}{\partial r} r' = q'r + qr' = 0. \tag{93}$$

Taking into account (41), this equation takes the form

$$\frac{q'}{q} = -\frac{r'}{r} = -\frac{1}{r}\frac{r}{R} = -\frac{1}{R}. \qquad (94)$$

In his solution

$$\ln q = -\ln R + \ln w \qquad (95)$$

the constant w we find from the matching condition on the surface:

$$q(R_b) = \frac{w}{R_b} = \frac{1}{r_g}, \quad w = \frac{R_b}{r_g}, \qquad (96)$$

which gives

$$q = \frac{w}{R} = \frac{R_b}{Rr_g}, \quad B = qr = \frac{R_b}{Rr_g} r. \qquad (97)$$

Substituting this into (92) we obtain

$$A' = \frac{r_{g,R}}{R} q = \frac{R}{R_b^2}, \quad A = \frac{1}{2}\frac{R^2}{R_b^2} + p, \qquad (98)$$

where the constant p is also found from the matching condition:

$$A(R_b) = \frac{1}{2} + p = 0, \quad p = -\frac{1}{2}. \qquad (99)$$

Finally, this gives the solution [5]:

$$A(R) = \frac{1}{2}\left(\frac{R^2}{R_b^2} - 1\right), \qquad (100)$$

$$y(R,\tau) = \frac{1}{2}\left(\frac{R^2}{R_b^2} - 1\right) + \frac{R_b}{Rr_g} r. \qquad (101)$$

The derivatives of y are equal to:

$$y' = \frac{R}{R_b^2}, \quad \dot{y} = \frac{R_b}{Rr_g}\dot{r} = -\frac{R^2}{R_b^2}\frac{1}{(r_{g,R} r)^{1/2}} \qquad (102)$$

and the diagonality condition (87) is satisfied.

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