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Four-index equations for gravitation and the gravitational energy-momentum tensor[1]

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Abstract

A new treatment of the gravitational energy on the basis of 4-index gravitational equations is reviewed. The gravitational energy for the Schwarzschild field is considered.

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Content

Introduction

A covariant physical characteristics of the gravitational field is the Riemann curvature tensor, and it is natural that the problems with the energy-momentum of gravitation can be solved if we can express the gravitational energy in terms of this tensor.

In the papers [1-2] a new generalized 4-index version of the Einstein equations with the Riemann tensor has been formulated, and the local energy-momentum tensors for the system of gravitation field and matter, linearly depending on the curvature tensor, have been constructed as 4-index tensors.

In the present paper some consequences of this treatment, including the calculation of the gravitational energy for a mass point, will be presented.

1. Four-index equations for the gravitational field

In the standard Einstein-Gilbert gravitational action one can add to the Ricci tensor or to the Riemann tensor arbitrary functions (tensors) with zero contractions:

$$R = g^{km}R_{km} = \frac{1}{2}(g^{km}g^{il} - g^{im}g^{kl})(R_{iklm} - \kappa V_{iklm}), \tag{1}$$

where $\kappa = 8\pi k / c^4$, and L_m is the matter Lagrangian, V_{iklm} has the same symmetry properties as R_{iklm} , and $g^{il}V_{iklm} = 0$.

So, we can start from the new action function:

[1] The preprint of the paper has been presented in 1999 (revised 2003): Zakir Z. arXiv:gr-qc/9906039.

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$$S = \frac{1}{2}\int d\Omega\sqrt{-g}\left[\frac{1}{2}(g^{km}g^{il} - g^{im}g^{kl})\left(-\frac{1}{\kappa}R_{iklm} + V_{iklm}\right) + L_m\right], \tag{2}$$

which is fully equivalent to the standard Einstein-Gilbert action function. Then we obtain for the variation of the action function [1]:

$$\delta S = -\frac{1}{2}\int d\Omega\sqrt{-g}\delta g^{km}g^{il}(G_{iklm} - T_{iklm}) = 0, \tag{3}$$

where $T_{iklm} = V_{iklm} + T^{(m)}_{iklm}$ and:

$$G_{iklm} = \frac{1}{\kappa}\left[R_{iklm} - \frac{1}{(d-1)(d-2)}(g_{il}g_{km} - g_{im}g_{kl})R\right], \tag{4}$$

$$\begin{aligned} T^{(m)}_{iklm} &= \frac{1}{(d-2)}(g_{km}T_{il} - g_{kl}T_{im} + g_{il}T_{km} - g_{im}T_{kl}) - \\ &- \frac{1}{(d-1)(d-2)}(g_{il}g_{km} - g_{im}g_{kl})T \end{aligned} \tag{5}$$

Here d is the spacetime dimensionality, and T_{iklm} has the same structure as the Riemann tensor having the representation:

$$\begin{aligned} R_{iklm} &= C_{iklm} + \frac{1}{(d-2)}(g_{km}R_{il} - g_{kl}R_{im} + g_{il}R_{km} - g_{im}R_{kl}) - \\ &- \frac{1}{(d-1)(d-2)}(g_{il}g_{km} - g_{im}g_{kl})R, \end{aligned} \tag{6}$$

where C_{iklm} is the Weyl tensor with zero 2-index contraction $g^{il}C_{iklm} = 0$. Thus, we obtain the equations:

$$g^{il}(G_{iklm} - T_{iklm}) = 0. \tag{7}$$

In a general case the expression in the parenthesis is not equal to zero for the arbitrary V_{iklm} and we cannot simply exclude the contractional factor g^{il} . However the tensor V_{iklm} has 10 independent components which is equal to the number of the Riemann tensor components in the vacuum ($T^{(m)}_{km} = 0$) where it is reduced to the Weyl tensor. Therefore, if in this case we choose the V_{iklm} as equal to:

$$\frac{1}{\kappa}G_{iklm} = \frac{1}{\kappa}C_{iklm} = V_{iklm}, \tag{8}$$

the equations hold identically for the solutions of the Einstein equations. Thus, we may write the 4-index equations for the gravitational field as [1]:

$$G_{iklm} = T_{iklm}. \tag{9}$$

We see that V_{iklm} can be considered as the 4-index energy-momentum density tensor for the gravitational field. Although its 2-index contraction vanish, in the 4-index form it allows one to determine a nonzero, local and positive defined energy-momentum tensor for the gravitational field.

The tensors G_{iklm} and T_{iklm} have the symmetry properties of the Riemann tensor and, therefore, we have 20 equations. The tensor G_{iklm} is a function of the metric tensor g_{ik} which has 6 independent components. The tensor $T^{(m)}_{iklm}$ has been combined from the ordinary energy-momentum tensor of the matter T_{ik} and it has 4 independent functions (the energy density ϵ and 3 components of the velocity). These 10 functions obey to 10 Einstein equations. The tensor V_{iklm} gives additional 10 independent components.

So, we have 20 equations for 20 independent functions. If we take solutions of the Einstein equations for the metrics and T_{ik} , then we have the additional 10 equations for 10 components of V_{iklm} . Therefore, the solutions of the Einstein equations exactly define all components of V_{iklm} and we can find V_{iklm} for the known standard metrics.

In the vacuum $T_{ik} = T = 0,\ R_{il} = R = 0$ and we have the equations Eq. (8). We see that in the vacuum the tensor V_{iklm} plays the role of the source for the empty spacetime curvature C_{iklm} .

The covariant derivatives of the 4-index tensors are also related as:

$$G^{i}_{klm\ ;i} = T^{i}_{klm;i}. \tag{10}$$

In the case $d = 4$ we have:

$$G^{i}_{.klm;i} = T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{,l} - g_{kl}T_{,m}), \tag{11}$$

$$T^{j(m)}_{klm;j} = \frac{1}{2}\left[T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{;l} - g_{kl}T_{;m})\right] = \frac{1}{2}G^{i}_{.klm;i}. \tag{12}$$

Then we obtain the relationship:

$$V^{j}_{klm;j} = G^{j}_{klm;j} - T^{j(m)}_{klm;j} = \frac{1}{2}G^{i}_{.klm;i}. \tag{13}$$

and, therefore,

$$V^{j}_{klm;j} = T^{j(m)}_{klm;j} \tag{14}$$

In the vacuum, therefore, there are local conservation laws:

$$G^{j}_{\cdot klm;j} = V^{j}_{klm;j} = 0. \tag{15}$$

The integral energy-momentum tensor for the system of matter and gravitational field can be defined as:

$$P^{i}_{lm} = \int dS_k T^{ik}_{..lm}. \tag{16}$$

On the hypersurface $x^0 = const$ we have:

$$P^{k}_{.lm} = \int d^3x\sqrt{-g}T^{0k}_{..lm} = \int d^3x\sqrt{-g}(T^{(m)0k}_{..lm} + V^{0k}_{..lm}). \tag{17}$$

The energy-momentum vector for matter can be obtained as: $P^i = P^{ik}_{.k}$. Finally, the 3-index integral energy-momentum of the gravitational field can be defined as:

$$P^{(g)i}_{.lm} = \int d^3x\sqrt{-g}V^{0i}_{..lm}. \tag{18}$$

2. The gravitational energy for the Schwarzschild field

Let us consider the energy of the Schwarzschild field with the line element:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{19}$$

where $r_g = 2Gm$ is the gravitational radius, and the components of the metric are: $g_{22} = -r^2$, $g_{33} = -r^2\sin^2\vartheta$, and:

$$g_{00} = g_{11}^{-1} = 1 - \frac{r_g}{r}. \tag{20}$$

We calculate the energy-momentum tensor:

$$V^{ik}_{lm} = \frac{1}{\kappa}R^{ik}_{lm} \tag{21}$$

for this solution of the Einstein equations. Nonzero components of the $V_{iklm} = R_{iklm} / \kappa$ with this metric are:

$$\begin{aligned} V_{0101} &= \frac{r_g}{\kappa r^3} = -V(r)g_{00}g_{11}, \\ V_{0202} &= -\frac{r_g(r - r_g)}{2\kappa r^2} = \frac{1}{2}V(r)g_{00}g_{22}, \\ V_{0303} &= -\frac{r_g(r - r_g)}{2\kappa r^2}\sin^2\vartheta = \frac{1}{2}V(r)g_{00}g_{33}, \end{aligned} \tag{22}$$

$$V_{1212} = \frac{r_g}{2\kappa(r - r_g)} = \frac{1}{2}V(r)g_{11}g_{22}, \tag{23}$$

$$V_{1313} = \frac{r_g\sin^2\vartheta}{2\kappa(r - r_g)} = \frac{1}{2}V(r)g_{11}g_{33}, \tag{24}$$

$$V_{2323} = -\frac{r_g r}{\kappa}\sin^2\vartheta = -V(r)g_{22}g_{33}, \tag{25}$$

where

$$V(r) = \frac{r_g}{\kappa r^3} = \frac{m}{4\pi r^3} = -\frac{m}{8\pi}\frac{\partial}{\partial r}(r^{-2}). \tag{26}$$

We see that the 2-index contraction of this tensor vanishes:

$$V_{il} = g^{km}V_{iklm} = g_{il}[-V(r) + \frac{1}{2}V(r) + \frac{1}{2}V(r)] = 0. \tag{27}$$

The physical components of the gravitational energy-momentum tensor $V^{ik}_{..lm} = g^{ip}g^{kq}V_{pqlm}$ are:

$$V^{01}_{..01} = V^{10}_{..10} = V^{23}_{..23} = V^{32}_{..32} = -V(r), \tag{28}$$

$$V^{02}_{..02} = V^{20}_{..20} = V^{03}_{..03} = V^{30}_{..30} = V^{12}_{..12} = \\ = V^{21}_{..21} = V^{13}_{..13} = V^{31}_{..31} = \frac{1}{2}V(r). \tag{29}$$

They allow us to calculate one of components of integral gravitational energy-momentum around the static spherical source as:

$$cP^{(g)1}_{.01} = \int dS_0 \sqrt{-g} V^{01}_{..01} = \int dS_0 \sqrt{-g}[-V(r)]. \tag{30}$$

The spatial volume integral can be represented as a spatial surface integral and we obtain:

$$cP^{(g)1}_{.01} = \frac{m}{8\pi}\int dS_0 \sqrt{-g}\,\frac{\partial}{\partial r}(r^{-2}) = \frac{m}{8\pi}\int df_{0r} r^{-2} = \frac{1}{2} n_r m, \tag{31}$$

where $df_{0r} = n_r r^2 do$ is 2-dimensional surface element with the normal vector n_r directed along r.

References

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