

## Four-index energy-momentum tensors for gravitation and matter<sup>1</sup>

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### Abstract

The 4-index energy-momentum tensors for gravitation and matter are analyzed on the basis of new equations for the gravitational field with the Riemann tensor. Some properties of such defined gravitational energy are discussed.

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Key words: gravitational energy, curvature tensor, vacuum energy

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### Introduction

In general relativity a true and covariant characteristic of the gravitational field is the Riemann curvature tensor  $R_{iklm}$ , but the field equations contain only the Ricci tensor, vanishing in the vacuum. The Weyl tensor  $C_{iklm}$  which is a nonvanishing in the vacuum pure 4-index part of  $R_{iklm}$ , disappear at 2-index contraction [1]. This fact was leading to the problems with the definition of the energy-momentum for the gravitational field.

In the paper [1] a new generalized version of the Einstein equations with the Riemann tensor has been formulated. It has been shown that the 4-index energy-momentum tensors for gravitation and matter can be constructed. In the present paper the structure of new gravitational equations and properties of 4-index energy-momentum tensors will be discussed.

#### 1. The action function and field equations with the Riemann tensor

We started from the standard Einstein-Gilbert action for the gravitational field:

$$\begin{aligned}
 S &= \frac{1}{2} \int d\Omega \sqrt{-g} \left( -\frac{1}{\kappa} R + L \right) = \\
 &= -\frac{1}{2} \int d\Omega \sqrt{-g} \left[ \frac{1}{2\kappa} (g^{il} g^{km} - g^{im} g^{kl}) R_{iklm} - L \right],
 \end{aligned} \tag{1}$$

where  $\kappa = 8\pi k / c^4$ . We perform the variational procedure so that  $R_{iklm}$  preserves its 4-index form. The result of such variation is [1]:

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<sup>1</sup> The preprint of the paper has been presented in 1999 (revised 2003): Zakir Z. [arXiv:gr-qc/9905036](https://arxiv.org/abs/gr-qc/9905036).

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$$\delta S_g = -\frac{1}{2} \int d\Omega \sqrt{-g} \delta g^{km} g^{il} [G_{iklm} - T_{iklm}] = 0. \quad (2)$$

Here new 4-index tensors are defined as ( $d = 4$ ):

$$\begin{aligned} G_{iklm} &= \frac{1}{\kappa} \left[ R_{iklm} - \frac{1}{6} (g_{il} g_{km} - g_{im} g_{kl}) R \right], \\ T_{iklm} &= V_{iklm} + \frac{1}{2} (g_{km} T_{il} - g_{kl} T_{im} + g_{il} T_{km} - g_{im} T_{kl}) - \\ &\quad - \frac{1}{6} (g_{il} g_{km} - g_{im} g_{kl}) T, \end{aligned} \quad (3)$$

where  $V_{iklm}$ , having the property  $g^{il} V_{iklm} = 0$  and which does not vanish in the vacuum around the source, can be identified by the required energy-momentum density tensor for gravitational field. The field equations in the general case are:

$$G_{iklm} = T_{iklm}. \quad (4)$$

The tensors  $G_{iklm}$  and  $T_{iklm}$  have the symmetry properties as the Riemann tensor and therefore we have 20 equations. The tensor  $G_{iklm}$  is a function of the metric tensor  $g_{ik}$  which has 6 independent components. The tensor  $T_{iklm}^{(m)}$  is combined from the ordinary energy-momentum tensor of the matter  $T_{ik}$  and it has 4 independent functions (the energy density  $\epsilon$  and 3 components of the velocity). These 10 functions are solutions of 10 Einstein equations (6 for independent components of the metric and 4 for independent components of  $T_{ik}$ ). The new term  $V_{iklm}$  has 10 independent components.

So, we have 20 equations for 20 independent functions. If we take solutions of the Einstein equations for some metric and  $T_{ik}$ , then we have an additional 10 equations for 10 components of  $V_{iklm}$ . This means that the solutions of the Einstein equations exactly define all components of  $V_{iklm}$  and we can find  $V_{iklm}$  for some standard metric. But if we have some model of the vacuum and calculate  $V_{iklm}$  in this model, then we have 10 equations for 10 unknown components of the metric  $g_{ik}$  and  $T_{ik}$ . In the vacuum  $G_{iklm}$  contains only the Weyl tensor  $C_{iklm}$  and we have the equations for the gravitational field.

## 2. The energy-momentum conservation for the system of gravitation and matter

The Riemann tensor can be represented as:

$$\begin{aligned} R_{iklm} &= C_{iklm} + \frac{1}{2} (g_{km} R_{il} - g_{kl} R_{im} + g_{il} R_{km} - g_{im} R_{kl}) - \\ &\quad - \frac{1}{6} (g_{il} g_{km} - g_{im} g_{kl}) R \end{aligned} \quad (5)$$

where  $C_{iklm}$  is the Weyl tensor with zero 2-index contraction  $g^{il} C_{iklm} = 0$ . In the vacuum  $T_{ik} = T = 0$ ,  $R_{il} = R = 0$ , and we have:

$$\frac{1}{\kappa} C_{iklm} = V_{iklm} \quad (6)$$

The covariant derivatives of these 4-index tensors are:

$$\begin{aligned} G^i{}_{.klm;i} &= \frac{1}{\kappa} \left[ R^i{}_{.klm;i} - \frac{1}{6} (g_{km} R_{,l} - g_{kl} R_{,m}) \right] = \\ &= T_{km;l} - T_{kl;m} - \frac{1}{3} (g_{km} T_{,l} - g_{kl} T_{,m}), \\ T^j{}_{klm;j} &= \frac{1}{2} \left[ T_{km;l} - T_{kl;m} - \frac{1}{3} (g_{km} T_{,l} - g_{kl} T_{,m}) \right] = \frac{1}{2} G^i{}_{.klm;i}. \end{aligned} \quad (7)$$

Then we obtain the relationship:

$$V^j{}_{klm;j} = G^j{}_{klm;j} - T^j{}_{klm;j} = \frac{1}{2} G^i{}_{.klm;i}. \quad (8)$$

and, therefore,

$$V^j{}_{klm;j} = T^j{}_{klm;j} \quad (9)$$

In the vacuum, therefore, there are local conservation laws:

$$G^j{}_{.klm;j} = V^j{}_{klm;j} = 0. \quad (10)$$

The integral energy-momentum tensors for matter and gravity can be defined as:

$$\begin{aligned} P^j{}_{ikl} &= \int dS_n \sqrt{-g} T^j{}_{ikl}{}^{(m)}, \\ V^j{}_{ikl} &= \int dS_n \sqrt{-g} V^j{}_{ikl}. \end{aligned} \quad (11)$$

In the hypersurface  $x^0 = const$  we have:

$$\begin{aligned} P^j{}_{ikl} &= \int d^3x \sqrt{-g} T^j{}_{ikl}{}^{(m)}, \\ V^j{}_{ikl} &= \int d^3x \sqrt{-g} V^j{}_{ikl}. \end{aligned} \quad (12)$$

We can construct the angular momentum tensor for the gravitational field as:

$$M^{iklm} = \int dS_n \sqrt{-g} (x^m V^{nikl} - x^i V^{nmkl}). \quad (13)$$

This 4-index tensor can be interpreted as *the angular momentum tensor* for the gravitational field.

### 3. Comparison with pseudotensor and Hamiltonian approaches

The pseudotensor  $t_{ik}$ , for example, in the Landau-Lifshitz version, has been defined as a part of the Einstein tensor:

$$G^{ik} = \frac{1}{(-g)} \frac{\partial \psi^{ikl}}{\partial x^l} - t^{ik}. \quad (14)$$

Such separation leads to the conservation of the sum of  $t^{ik}$  and  $T^{ik}$  :

$$\int dS_k (-g) (T^{ik} + t^{ik}) = \int dS_k \frac{\partial \psi^{ikl}}{\partial x^l}, \quad (15)$$

and the right hand term then can be interpreted as a total energy of the system.

Thus, if we want to work with a localizable and tensor form of the gravitational energy-momentum in the vacuum, we can take the Weyl tensor, which has zero 2-index

contraction. If we want to work with non-zero 2-index form of the gravitational energy-momentum, we take the pseudotensors or the Hamiltonians, which are non-localizable and non-covariant.

#### 4. The geodesic deviation and measurements of the gravitational energy

The equation of geodesic deviation:

$$\frac{D^2 \eta^i}{ds^2} = R_{klm}^i u^k u^l \eta^m \quad (16)$$

we rewrite in vacuum as:

$$\frac{D^2 \eta^i}{ds^2} = C_{klm}^i u^k u^l \eta^m = \kappa V_{klm}^i u^k u^l \eta^m. \quad (17)$$

Therefore, we conclude that the measurements of the geodesic deviations are exactly the measurements of the 4-index energy-momentum tensor of gravitational field  $V_{klm}^i$ .

In [2] the energies for standard metrics and some consequences of the proposed treatment of the gravitational energy will be considered.

#### References

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